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Multi-soliton asymptotics for the SU(N) sigma model

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Abstract. The asymptotic behaviour of the multi-soliton solution for the SU(N) sigma model is analysed. It is shown that if the diagonal matrices A_0 and $B_0 \in \mathrm{su}(n)$ defining the vacuum solution are non-degenerate (their diagonal elements are all distinct), the multi-soliton solution is asymptotically diagonal.

1. Introduction

In the past many articles have been devoted to finding solutions to certain non-linear equations admitting Zakharov-Shabat or Lax pair representations (e.g. see Zakharov and Mïkhailov 1978a). These equations are characterised by interesting properties such as an infinity of conservation laws and soliton solutions (Chau Wang 1980). A soliton solution has the property of becoming asymptotically free of any interaction (Miura 1976). Soliton solutions can be obtained through the use of Bäcklund transformations which give a non-trivial solution built upon a vacuum solution or other background solution (Chau Wang 1980).

In Ogielski *et al* (1980) a Bäcklund transformation is given for the two-dimensional SU(N) principal sigma model associated with the field equations (Zakharov and Mïkhailov 1978b):

$$A_{0\eta} + B_{0\xi} = 0$$

$$A_{0\eta} - B_{0\xi} + [A_0, B_0] = 0.$$

This gives the solution $\psi_0(\lambda)$ defined by Harnad *et al* (1984a) which satisfies the system of matrix equations:

$$\psi_{0\xi} = \frac{A_0\psi_0}{1+\lambda} \qquad \text{and} \qquad \psi_{0\eta} = \frac{B_0\psi_0}{1-\lambda} \tag{1.1}$$

where ξ and η are the light-cone coordinates defined by

$$\xi = \frac{x+t}{2} \qquad \eta = \frac{x-t}{2}.$$

An n-soliton solution can be obtained from the vacuum solution by applying a sequence of 'n' Bäcklund transformations, which gives rise to a nonlinear superposition of n-soliton solutions (Harnad *et al* 1980a, Zakharov and Mïkhailov 1978a, Zakharov and Shabat 1979). The aim of this paper is to show that this nonlinearity disappears asymptotically, giving rise to a genuine superposition of solitons with disjoint support.

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2. Asymptotic behaviour of the multi-soliton solution

Following the notation of Harnad *et al* (1984a) one wishes to express explicitly the Bäcklund transformation in terms of the initial solution and the initial value parameters $\{\lambda_1, m_1\}$. The projector P_1 (henceforth P) itself must be expressed in terms of these parameters since the Bäcklund transformation is effected by it.

The sequence of solutions defined by Harnad *et al* (1984a) (see also Zakharov and Mikhailov 1978a, Zakharov and Shabat 1979) shows that only $\psi_0(\lambda)$ has to be integrated explicitly, the other solutions being given by an algebraic computation. If one supposes that ψ_0 is now known, $\psi_1(\lambda)$ can be determined by the following computation (Harnad *et al* 1984a):

$$\begin{aligned} &M_1=\psi_0(\bar\lambda_1)m_1 \qquad X_1(\lambda)=\left(I+\frac{\lambda_1-\bar\lambda_1}{\lambda-\lambda_1}P\right) \\ &P=M_1(M_1^\dagger M_1)^{-1}M_1^\dagger \qquad m_1\in C^{n\times r_1} \qquad r_1=rkP \\ &\psi_1(\lambda)=X_1(\lambda)\psi_0(\lambda). \end{aligned} \tag{2.1}$$

Furthermore let $d_{st}\in C$ be the components of m_1, $1\leqslant s\leqslant n$, $1\leqslant t\leqslant r_1$. One chooses the matrices A_0 and B_0 to represent a vacuum solution, i.e. a solution such that $A_0\equiv g_\xi g^{-1}$ and $B_0\equiv g_\eta g^{-1}$ commute and depend, respectively, on ξ and η only (Saint-Aubin 1982). An appropriate choice is

$$A_0=\mathrm{i}\,\mathrm{diag}(a_1\ldots a_n) \qquad \text{and} \qquad B_0=\mathrm{i}\,\mathrm{diag}(b_1\ldots b_n).$$

Since they are constant and commute, they can be simultaneously diagonalised by a constant matrix. Without loss of generality, it can be assumed that this has already been done.

Since A_0 and B_0 belong to su(n)

$$\sum_{i=1}^n a_i=\sum_{i=1}^n b_i=0 \qquad a_i,\, b_i\in R.$$

The integration of equation (1.1) gives

$$\psi_0(\lambda)=\mathrm{diag}\left(\exp\mathrm{i}\left(\frac{a_1\xi}{1+\lambda}+\frac{b_1\eta}{1-\lambda}\right),\ldots,\exp\mathrm{i}\left(\frac{a_n\xi}{1+\lambda}+\frac{b_n\eta}{1-\lambda}\right)\right). \tag{2.2}$$

Then $\psi_0(\lambda)$ is completely determined; M_1 may now be found, and hence P. The expression for M_1 in terms of the initial parameters and the variables ξ and η is

$$[M_1]_{lm}=[\psi_0(\bar\lambda_1)m_1]_{lm}=d_{lm}\exp\mathrm{i}\left(\frac{a_l\xi}{1+\bar\lambda_1}+\frac{b_l\eta}{1-\bar\lambda_1}\right)\equiv f_{lm} \tag{2.3}$$

where $\lambda\equiv\lambda_1$, $1\leqslant l\leqslant n$, $1\leqslant m\leqslant r_1$. Let us denote the scalar product as

$$\langle f_i|f_j\rangle=\sum_{l=1}^n \bar f_{li}f_{lj} \tag{2.4}$$

where $1\leqslant i,\, j\leqslant r_1$. From equation (2.3), the projector can be written in terms of its components

$$P_{\mu\nu}=f_{\mu a}[(f^\dagger f)^{-1}f^\dagger]_{a\nu} \tag{2.5}$$

where $f_{\mu a}=(f_{\mu 1},\ldots,f_{\mu r_1})$. Substituting in equation (2.5) the scalar product defined by equation (2.4) gives, after rearranging the terms,

$$P_{\mu\nu}=\frac{\sum_{m=1}^{r_1}\varepsilon^{i_1\ldots m\ldots i_{r_1}}\langle f_1|f_{i_1}\rangle\ldots(f_{\mu m}\bar{f}_{\nu i_m})\ldots\langle f_{r_1}|f_{i_{r_1}}\rangle}{\varepsilon^{i_1\ldots i_{r_1}}\langle f_1|f_{i_1}\rangle\ldots\langle f_{r_1}|f_{i_{r_1}}\rangle}. \tag{2.6}$$

Note that the term $\langle f_m|f_{i_m}\rangle$ does not appear in equation (2.6) but it is replaced by $(f_{\mu m}\bar{f}_{\nu i_m})$.

The projector $P_{\mu\nu}$ can be expressed in terms of initial parameters using equation (2.3):

$$f_{\mu m}\bar{f}_{\nu j}=d_{\mu m}\bar{d}_{\nu j}\exp(\mathrm{i}\theta_{\mu\nu})\exp(\gamma_{\mu\nu}t+\beta_{\mu\nu}x) \tag{2.7}$$

$$\langle f_i|f_j\rangle=\sum_{l=1}^{n}\bar{d}_{li}d_{lj}\exp(\gamma_{ll}t+\beta_{ll}x) \tag{2.8}$$

where the following quantities have been defined:

$$\theta_{\mu\nu}\equiv\left(\frac{(a_\nu-a_\mu)(1+\operatorname{Re}\lambda)\xi}{|1+\lambda|^2}+\frac{(b_\nu-b_\mu)(1-\operatorname{Re}\lambda)\eta}{|1-\lambda|^2}\right)\equiv-\theta_{\nu\mu}$$

$$\beta_{\mu\nu}\equiv\frac{-\operatorname{Im}\lambda}{2}\left(\frac{(a_\nu+a_\mu)}{|1+\lambda|^2}-\frac{(b_\nu+b_\mu)}{|1-\lambda|^2}\right) \tag{2.9a}$$

$$\gamma_{\mu\nu}\equiv\frac{-\operatorname{Im}\lambda}{2}\left(\frac{(a_\nu+a_\mu)}{|1+\lambda|^2}+\frac{(b_\nu+b_\mu)}{|1-\lambda|^2}\right). \tag{2.9b}$$

The soliton region is described by means of the projector on the form (2.6). This region is determined by the values of x and t giving a non-negligible contribution to the summation of the indices m and i_m in equation (2.6). Of course, this region depends upon the different values of $\gamma_{\mu\nu}$ and $\beta_{\mu\nu}$ and thus of A_0 and B_0.

Now to continue further it is essential to assume that A_0 and B_0 have been chosen such that there is no degeneracy, in other words $a_\nu\neq a_\mu$, $b_\nu\neq b_\mu$ for any pair (μ, ν, $\mu\neq\nu$). This implies

$$\gamma_{\mu\mu}<\gamma_{\mu\nu}<\gamma_{\nu\nu}\qquad\text{or}\qquad\gamma_{\nu\nu}<\gamma_{\mu\nu}<\gamma_{\mu\mu} \tag{2.10}$$

and similarly for $\beta_{\mu\nu}$. The explicit form of the projector in terms of the quantities (2.9a, b) is easily found by substituting equations (2.7) and (2.9) into equation (2.6) giving:

$$P_{\mu\nu}=N_{\mu\nu}/\Delta$$

where

$$\begin{aligned}N_{\mu\nu}=\Bigg[\sum_{m=1}^{r_1}\varepsilon^{i_1\ldots i_{r_1}}&\left(\sum_{l_1=1}^{n}\bar{d}_{l_1 1}d_{l_1 i_1}\exp(\gamma_{l_1l_1}t+\beta_{l_1l_1}x)\right)\ldots\\&\times[d_{\mu m}\bar{d}_{\nu i_m}\exp(\mathrm{i}\theta_{\mu\nu})\exp(\gamma_{\mu\nu}t+\beta_{\mu\nu}x)]\ldots\\&\times\left(\sum_{l_{r_1}=1}^{n}\bar{d}_{l_{r_1}r_1}d_{l_{r_1}i_{r_1}}\exp(\gamma_{l_{r_1}l_{r_1}}t+\beta_{l_{r_1}l_{r_1}}x)\right)\Bigg]\\\Delta=\varepsilon^{i_1\ldots i_{r_1}}&\left(\sum_{l_1=1}^{n}\bar{d}_{l_1 1}d_{l_1 i_1}\exp(\gamma_{l_1l_1}t+\beta_{l_1l_1}x)\right)\ldots\\&\times\left(\sum_{l_{r_1}=1}^{n}\bar{d}_{l_{r_1}r_1}d_{l_{r_1}i_{r_1}}\exp(\gamma_{l_{r_1}l_{r_1}}t+\beta_{l_{r_1}l_{r_1}}x)\right).\end{aligned} \tag{2.11}$$

Since by hypothesis the eigenvalues of A_0 and B_0 are distinct, it is possible by virtue of equation (2.10) to order the γ_{ii}. Without loss of generality the following choice can be assumed:

$$\gamma_{11} < \gamma_{22} < \ldots < \gamma_{nn}. \tag{2.12}$$

Moreover, assuming that the space coordinate x is a fixed quantity, the asymptotic behaviour of $P_{\mu\nu}$ when $t \to +\infty$ is sought. According to the order in which the γ_{ii} are classified, the dominant term in the numerator $N_{\mu\nu}$ is the one in which all l_j but one take the value n since the terms containing just γ_{nn} do not contribute to the sum. Thus the numerator behaves like

$$\exp[(r_1-2)\gamma_{nn}+\gamma_{n-1,n-1}+\gamma_{\mu\nu}]t+\mathrm{O}(\exp[(r_1-3)\gamma_{nn}+2\gamma_{n-1,n-1}+\gamma_{\mu\nu}]t) \qquad \text{if } \mu \neq \nu$$

for $t \to +\infty$. The same argument shows that the dominant term in the denominator is

$$\exp\{[(r_1-1)\gamma_{nn}+\gamma_{n-1,n-1}]t\}+\mathrm{O}(\exp\{[(r_1-2)\gamma_{nn}+2\gamma_{n-1,n-1}]\}) \qquad \text{if } \mu \neq \nu$$

for $t \to +\infty$. Therefore the limiting form of $P_{\mu\nu}$ is

$$\lim_{t\to+\infty} P_{\mu\nu} = \lim_{t\to+\infty} \exp -(\gamma_{nn}-\gamma_{\mu\nu})t = 0 \qquad \text{for } \mu \neq \nu \tag{2.13}$$

where the fact that γ_{nn} is always greater than $\gamma_{\mu\nu}$ has been used. This argument fails for $\mu = \nu$ and thus no conclusions can be drawn for this case. Obviously a similar result holds for $t \to -\infty$ but the dominant term is γ_{11} instead of γ_{nn} (if the order (2.12) is kept for the $\gamma_{\mu\mu}$). Thus the non-diagonal elements $P_{\mu\nu}$ have the following behaviour:

$$\lim_{t\to\mp\infty} P_{\mu\nu} = 0 \qquad \text{if } \mu \neq \nu$$

for the non-degenerate matrices A_0 and B_0. Hence the only possible non-vanishing terms of the projector are the diagonal ones. Since the eigenvalues of a projector are 0 or 1, the following result can be stated:

$$\lim_{t\to\mp\infty} P_{\mu\mu} = 0 \text{ or } 1 \qquad \text{where } 1 < \mu < n. \tag{2.14}$$

By a similar estimate the same result for fixed t and $x \to \mp\infty$ is obtained, namely an asymptotically diagonal P. More generally, according to the equations (2.6) and (2.7), one can change $(\gamma_{\mu\nu}, \beta_{\mu\nu}) \to (\gamma'_{\mu\nu}, \beta'_{\mu\nu})$ which is equivalent to changing $(x, t) \to (x', t')$ showing that a result similar to (2.14) holds asymptotically in any spacetime region except the soliton region. The soliton region will be the subject of the next section. The main conclusion of this section is:

In the case of no degeneracy in the matrices A_0 and B_0, the soliton solution obtained by Bäcklund transformation for the SU(N) principal sigma model becomes asymptotically diagonal.

3. Soliton region

The results of the previous section do not hold in the soliton region since

$$\gamma_{\mu\mu}t+\beta_{\mu\mu}x \to 0 \qquad \gamma_{\nu\nu}t+\beta_{\nu\nu}x \to 0 \tag{3.1a}$$

or

$$\gamma_{\mu\nu}t+\beta_{\mu\nu}x \to 0 \tag{3.1b}$$

in that region, implying that the term with the indices (μ, μ), (ν, ν) and (μ, ν) give a similar contribution in equation (2.11). If equations (3.1*a*, *b*) are satisfied, the diagonal and non-diagonal elements $P_{\mu\mu}$, $P_{\nu\nu}$, $P_{\mu\nu}$ are comparable in magnitude. A numerical example with degeneracy will be presented below. First let us study in more detail the soliton region, i.e. the consequences of equations (2.1*a*, *b*). Equation (2.6) can be rewritten as

$$P_{\mu\nu} = \sum_{m=1}^{r_1} \frac{\sum_{i_m=1}^{r_1} f_{\mu m} \bar{f}_{\nu i_m} \Delta_{m i_m}}{\sum_{k=1}^{r_1} \langle f_m | f_k \rangle \Delta_{mk}} \tag{3.2}$$

where Δ_{ij} is the minor determinant of the corresponding element $f_{\mu i}\bar{f}_{\nu j}$ or $\langle f_i | f_j \rangle$. It is easily checked that equation (3.2) can also take the form:

$$P_{\mu\nu} = \sum_{l,m=1}^{r_1} \left(\sum_{k=1}^{r_1} \frac{\Delta_{mk}}{\Delta_{ml}} [K_{\mu\nu}^{klm} \cosh \Omega_{\mu\nu}(x_{0\mu\nu}^{klm} + x + v_{\mu\nu}t) + S_{\mu\nu}^{klm}] \right)^{-1} \tag{3.3}$$

with the following definitions:

$$K_{\mu\nu}^{klm} \equiv 2\left(\frac{\bar{d}_{\mu k} d_{\mu m} \bar{d}_{\nu l}}{\bar{d}_{\nu l} d_{\nu m} \bar{d}_{\nu k}} \right) \exp \mathrm{i}(\theta_{\mu\nu} + \varphi_{\mu\nu}^{klm})$$

$$\varphi_{\mu\nu}^{klm} \equiv \mathrm{Im}\left(\frac{\bar{d}_{\mu k} d_{\mu m} \bar{d}_{\nu l}}{\bar{d}_{\nu l} d_{\nu m} \bar{d}_{\nu k}} \right) \left[2\, \mathrm{Re}\left(\frac{\bar{d}_{\mu k} d_{\mu m} \bar{d}_{\nu l}}{\bar{d}_{\nu l} d_{\nu m} \bar{d}_{\nu k}} \right) \right]^{-1} \qquad S_{\mu\nu}^{klm} \equiv \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} \frac{f_{jm} \bar{f}_{jk}}{f_{\mu m} \bar{f}_{\nu l}}$$

$$x_{0\mu\nu}^{klm} \equiv \frac{\ln(r_{\mu\nu}^{klm})}{2\Omega_{\mu\nu}} \qquad r_{\mu\nu}^{klm} \equiv \left| \frac{\bar{d}_{\mu k} d_{\mu m} \bar{d}_{\nu l}}{\bar{d}_{\nu l} d_{\nu m} \bar{d}_{\nu k}} \right| \tag{3.4}$$

$$\tau_{\mu\mu} \equiv \gamma_{\mu\mu} - \gamma_{\mu\nu} \qquad \Omega_{\mu\nu} \equiv \beta_{\mu\mu} - \beta_{\mu\nu}$$

$$v_{\mu\nu} = \frac{\tau_{\mu\nu}}{\Omega_{\mu\nu}} = \left(\frac{(a_\nu - a_\mu)}{|1+\lambda|^2} + \frac{(b_\nu - b_\mu)}{|1-\lambda|^2} \right) \left(\frac{(a_\nu - a_\mu)}{|1+\lambda|^2} - \frac{(b_\nu - b_\mu)}{|1-\lambda|^2} \right)^{-1}. \tag{3.5}$$

This last formula for $v_{\mu\nu}$ can be interpreted as being the soliton velocity, and coincides with the one found by Zakharov and Mikhailov (1978b) in the particular case of SU(3) and a projector of rank one, except for differences in sign resulting from a different choice of light-cone coordinates. The soliton condition (3.1*b*) can be rewritten:

$$x + v_{\mu\nu} \to 0. \tag{3.6}$$

The function $C_1(x, t)$ can be introduced as follows:

$$C_1(x, t) = \exp \mathrm{i}\left(\frac{a_1 \xi}{1+\bar{\lambda}} + \frac{b_1 \eta}{1-\bar{\lambda}} \right)$$

and it is assumed that for an asymptotic region they verify:

$$|C_1| > |C_2| > \ldots > |C_n|. \tag{3.7}$$

Therefore, asymptotically equation (2.4) can be rewritten:

$$\langle f_i | f_k \rangle = |C_1|^2 D_{ik} + E_{ik} \tag{3.8}$$

where

$$E_{ik} = \sum_{j=2}^{n} |C_j|^2 \bar{d}_{ji} d_{jk} \qquad \text{and} \qquad D_{ik} \equiv \bar{d}_{1i} d_{1k}$$

which implies that $E_{ik} \ll |C_1|^2 D_{ik}$, in view of equation (3.7). On the other hand, from equation (2.7) the quantity $S_{\mu\nu}^{klm}$ is

$$S_{\mu\nu}^{klm} \equiv \sum_{\substack{s=1 \\ s \neq \mu,\nu}}^{n} \frac{\bar{d}_{sm} d_{sk} \exp(\gamma_{ss} t + \beta_{ss} x)}{d_{\mu m} \bar{d}_{\nu l} \exp(\mathrm{i}\theta_{\mu\nu}) \exp(\gamma_{\mu\nu} t + \beta_{\mu\nu} x)}. \tag{3.9}$$

Therefore if it is assumed that there are no s' such that

$$\beta_{\mu\nu}\gamma_{s's'} \neq \beta_{s's'}\gamma_{\mu\nu} \tag{3.10}$$

then all the terms

$$\exp(\gamma_{ss} t + \beta_{ss} x)$$

contained in equation (3.9) vanish or becomes infinite asymptotically. The case where the quantity defined by equation (3.9) becomes infinite is not of interest since $P_{\mu\nu}$ vanishes. Therefore the interesting case is when the quantity defined by equation (3.9) is vanishing, giving, Instead of (3.3),

$$P_{\mu\nu} = \sum_{l,m=1}^{r_1} \left(\sum_{k=1}^{r_1} \frac{\Delta_{mk}}{\Delta_{ml}} [K_{\mu\nu}^{klm} \cosh \Omega_{\mu\nu}(x_{0\mu\nu}^{klm} + x + v_{\mu\nu} t)] \right)^{-1} \tag{3.11}$$

for the projector in the soliton region. It is interesting to notice that if as $t \to +\infty$ ($t \to -\infty$) we obtain a soliton, as $t \to -\infty$ ($t \to +\infty$), $P_{\mu\nu}$ is vanishing. The case where the condition equation (3.10) is satisfied is equivalent to equation (3.11) since this is, in fact, just as if there were no summation on $s = s'$ in equation (3.9).

Let Δ'_{ik} denote the minor of the element D_{ij}; the ratio of Δ_{mk} to Δ_{ml} is

$$\frac{\Delta_{ik}}{\Delta_{lm}} \approx \frac{\Delta'_{ik} + \mathrm{O}(|C_2|^2 |C_1|^{-2})}{\Delta'_{lm} + \mathrm{O}(|C_2|^2 |C_1|^{-2})} \approx \frac{\Delta'_{ik}}{\Delta'_{lm}} + \mathrm{O}(|C_2|^2 |C_1|^{-2}) \tag{3.12}$$

where the form (3.8) for the scalar product has been used. Since D is a constant matrix the ratio of the two determinants in equation (3.12) must be bounded. Therefore, since the projector is non-diagonal, a soliton will appear in this spacetime region.

4. Degeneracy of the eigenvalues a_i and b_i in A_0 and B_0 of the vacuum solution

As mentioned, the above results hold because of the assumption of non-degeneracy for A_0 and B_0. A counter-example will show that the solution g may be non-diagonal in case of degeneracy. Let the counter-example be in SU(3):

$$A_0 = \mathrm{diag}(a_1, a_2, a_3) \qquad \text{and} \qquad B_0 = \mathrm{diag}(b_1, b_2, b_3)$$

where $A_0, B_0 \in \mathrm{su}(3)$. Let m be given by

$$m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and consider the definition:

$$\gamma \equiv \exp \mathrm{i}\left(\frac{a_3 \xi}{1+\lambda} + \frac{b_3 \eta}{1-\lambda} \right).$$

For simplicity let $a_2 = a_3$ and $b_2 = b_3$, from which the vacuum solution takes the form:

$$\psi_0(\bar{\lambda}) = \begin{pmatrix} \gamma^{-2} & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

The matrix M is

$$M = \psi_0(\bar{\lambda})m = \begin{pmatrix} \gamma^{-2} & \gamma^{-2} \\ 0 & \gamma \\ \gamma & 0 \end{pmatrix}.$$

Then, assuming that a_i, b_i are chosen such that $\gamma \to 0$ as $t \to +\infty$,

$$\lim_{t\to+\infty} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

This counter-example shows explicitly the privileged role played by the non-degenerate matrices A_0 and B_0 in the asymptotic behaviour of the soliton solutions.

5. Multi-soliton asymptotics

The asymptotic behaviour for one soliton generated after one Bäcklund transformation from the vacuum has been obtained. It is appropriate now to study the asymptotic behaviour of 'l' solitons generated by 'l' Bäcklund transformations. The generalisation to the multi-soliton case is easily obtained when the behaviour of the first Bäcklund transformation is known. Here the sequence of Bäcklund transformations of Harnad *et al* 1984a, c) is used, with the same notation. Schematically this sequence can be represented as

$$(A_0, B_0, \lambda_1) \to (A_1, B_1, \lambda_2) \to \ldots \to (A_{l-1}, B_{l-1}, \lambda_l)$$

which is obtained by the following calculation:

$$\tilde{M}_i = \psi_{i-1}(\bar{\lambda}_i)m_i \qquad X_i(\lambda) = \left(I + \frac{\lambda_i - \bar{\lambda}_i}{\lambda - \lambda_i} P_i\right)$$

$$P_i = \tilde{M}_i(\tilde{M}_i^\dagger \tilde{M}_i)^{-1}\tilde{M}_i^\dagger \qquad \tilde{M}_i \in C^{n\times r_i} \qquad r_i = rkP_i$$

$$\psi_i(\lambda) = X_i(\lambda)\psi_{i-1}(\lambda).$$

Since, as shown in the non-degenerate case, P becomes diagonal asymptotically so does $X_1(\lambda)$ and hence the solution ψ_1. The matrices A_1 and B_1 are also diagonal since they are defined by

$$A_1 \equiv \psi_{1,\xi}(\lambda = 0)\psi_1^{-1}(\lambda = 0) \qquad B_1 \equiv \psi_{1,\eta}(\lambda = 0)\psi_1^{-1}(\lambda = 0). \tag{5.1}$$

Harnad *et al* (1984b) obtain the following recurrence relation between the A_i and B_i:

$$A_i = X_i(\lambda = -1)A_{i-1}X_i^{-1}(\lambda = -1) \tag{5.2}$$

$$B_i = X_i(\lambda = -1)B_{i-1}X_i^{-1}(\lambda = -1) \tag{5.3}$$

showing that A_i and B_i have, respectively, the same spectrum as A_{i-1} and B_{i-1} and are therefore non-degenerate. In other words, the results (5.1)-(5.3) together imply that the asymptotic conditions satisfied before the Bäcklund transformations are still valid afterwards. Therefore the proof of the asymptotic behaviour of P_1 holds similarly for P_2 and so on. A question arises here for the soliton region of the first solution since in this region the projector is not diagonal and therefore the previous argument does not hold. What is the asymptotic solution in the soliton region of the first solution after the second Bäcklund transformation? This can be quickly answered by the use of the permutability theorem (Harnad *et al* 1984a). On the other hand, the second soliton has, as does the first, the appearance of an isolated soliton since it is built from a diagonal background. Moreover the solutions obtained by performing two Bäcklund transformations with λ_1 and λ_2 interchanged must be identical. The use of this property in the soliton region of the first solution makes possible the previous argument. In other words, in the region where the first solution is not diagonal, the second soliton is considered as being the first one since the final solution will be the same. Thus the following result can be stated:

The multi-soliton solutions for the SU(N) principal sigma model obtained from a non-degenerate vacuum solution after 'l' Bäcklund transformations become diagonal asymptotically.

This means that the nonlinearity disappears completely asymptotically and each soliton can be considered as isolated.

Now it will be shown by a numerical example of soliton solutions that the converse of the previous result is false. Indeed, contrary to the non-degenerate case the asymptotic behaviour is completely determined only with the explicit knowledge of the initial parameters. A numerical calculation is appropriate for this case since the asymptotic behaviour can change drastically when m is varied. Our aim is now to show this fact with an example for a solution in SU(4). It is solely for numerical convenience that SU(4) has been chosen. After integration this leads to one or two terms which are negligible compared to the other(s). The choice of SU(4) allows more freedom in selecting the elements of the matrices A_0, B_0 and m; this freedom is very useful in representing the postulated phenomenon. Consider the following values for A_0 and B_0:

$$A_0 = \mathrm{i}\ \mathrm{diag}(-0.45, -0.75, 0.6, 0.6) \tag{5.4a}$$

$$B_0 = \mathrm{i}\ \mathrm{diag}(1.2, 0.3, -0.75, -0.75). \tag{5.4b}$$

The values of m are chosen such that the rank of the projector is two and the first columns of the two matrices are distinct. The two different matrices m and m' here are

$$m = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad m' = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}. \tag{5.5}$$

The parameter λ of the Bäcklund transformation is

$$\lambda = \tfrac{1}{2} + (\sqrt{3}/2)\mathrm{i} \tag{5.6}$$

for both values of m. The solutions corresponding to m and m' are represented in figures 1 and 2, respectively. In all figures $x, t \in [-18, +18]$. Only the norms of $\bar{g}_{43} = g_{34}$, g_{44}, g_{33} are shown since these elements display the fact that they share the 'soliton

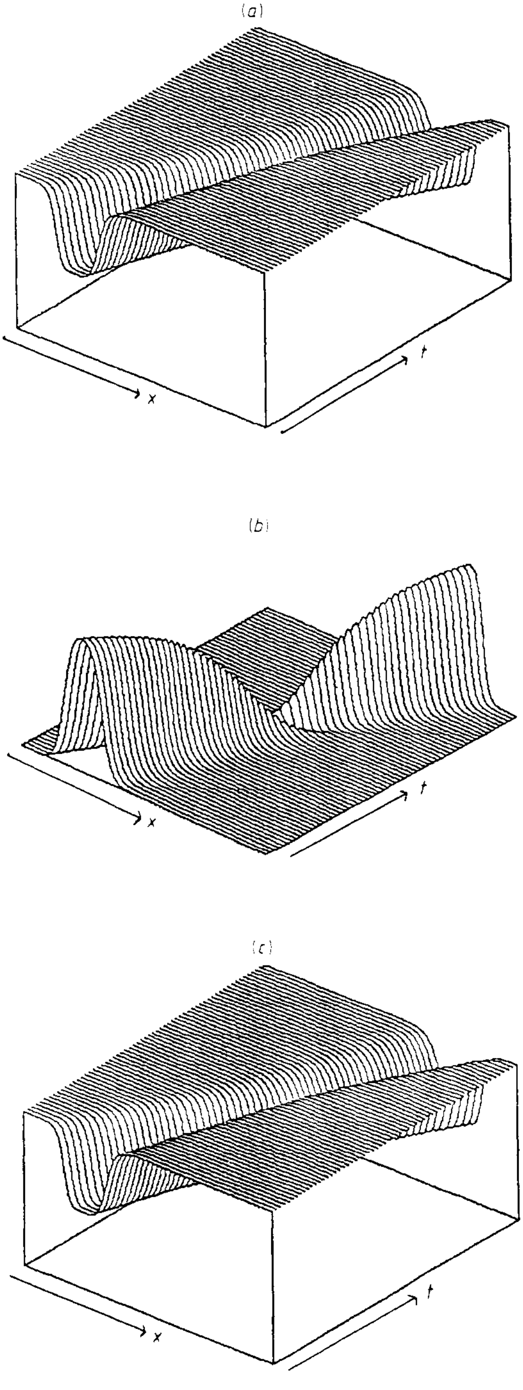

Figure 1. Norm of the elements g_{33}, $g_{34} = \overline{g_{43}}$, g_{44}, where $g \in SU(4)$ for a projector of rank 2 associated with the matrix m and parameter λ defined by equation (5.6) for degenerate matrices A_0 and B_0 given by equation (5.4*a*, *b*). (*a*) Norm of the element g_{33} associated with m, (*b*) Norm of the element $g_{34} = \overline{g_{43}}$ associated with m, (*c*) Norm of the element g_{44} associated with m.

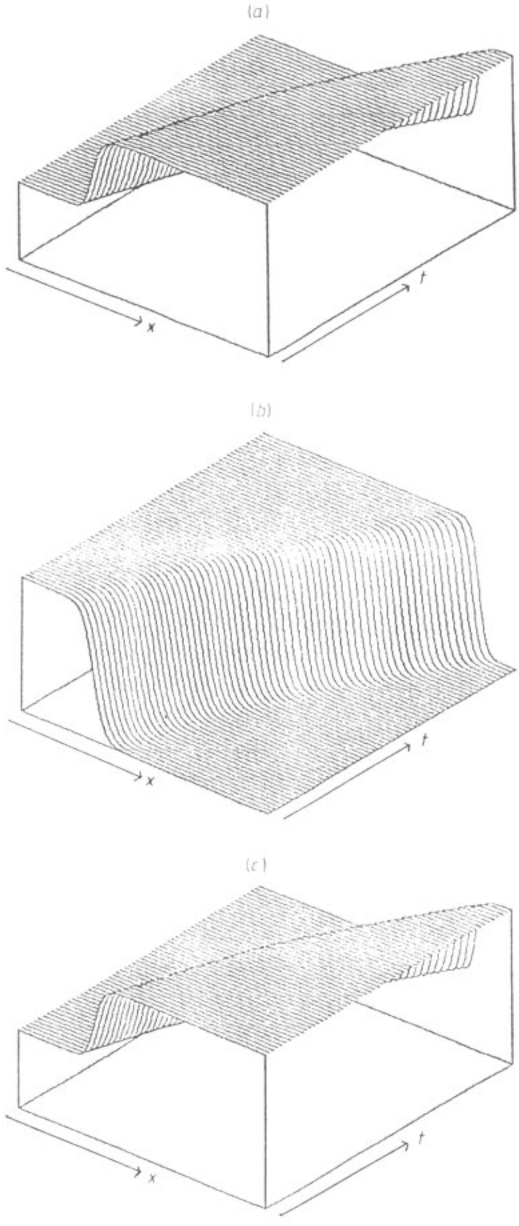

Figure 2. Norm of the elements g_{33}, $g_{34}=\overline{g_{43}}$, g_{44}, where $g \in SU(4)$ for a projector of rank 2 associated with the matrix m' and parameter λ defined by equation (5.6) for degenerate matrices A_0 and B_0 given by equation (5.4*a*, *b*). (*a*) Norm of the element g_{33} associated with m', (*b*) Norm of the element $g_{34}=\overline{g_{43}}$ associated with m', (*c*) Norm of the element g_{44} associated with m'.

energy' between them. These four elements of g have the subscripts of the non-distinct diagonal elements of A_0 and B_0. It is important to observe that even if the matrices A_0 and B_0 are degenerate, the solution with m is still diagonal. However the solution with m' is not diagonal, even through m and m' are only slightly different. From figures 1 and 2 it is clear that degeneracy is necessary but not sufficient to obtain a non-diagonal solution.

In conclusion, it is clear that for such a case an explicit calculation is required to determine the asymptotic behaviour of the solution.

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